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A Dynamical Theory of Brownian Motion for the Rayleigh Gas

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A dynamical theory of the Brownian motion is worked out for the Rayleigh gas and open problems of this theory are surveyed.

KEY WORDS: Diffusion limit; Brownian motion; mass and space-time scaling.

1. INTRODUCTION

Recent progress[1,14,17,18] has again raised hopes of understand the dynamical theory of Brownian motion for simplified models at least. In fact, Ref. 18 formulates such a theory for the Rayleigh gas, i.e., for the case when the Brownian particle interacts with an ideal gas (cf. Ref. 16). Moreover, mathematical results were also obtained supporting the theory.

The aim of the present paper is to explain the main ideas of the aforementioned paper in order to understand which open problems of the theory seem realistic and, roughly speaking, what kind of difficulties should be overcome in their solution.

To prepare the exposition of the theory (presented in Section 4), we give a mathematical formulation of the model in Section 2 and survey some previous results in Section 3. The main components of the sketchy proof outlined in Section 5 are explained in Sections 6 (the Markovian approximation) and 7 (the coupling). Finally, Section 8 surveys the most interesting open problems of the theory for both the one- and multi-dimensional cases (apart from this last point, we restrict ourselves to the 1D case).

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2. FORMULATION OF THE MODEL

A one-dimensional system of point particles consists of a tagged particle of mass M (the Brownian particle) interacting with an infinite ideal gas of particles of mass 1 (light particles). The dynamics of the system is governed by the laws of classical mechanics, assuming uniform motion plus elastic collisions between the Brownian particle and the light ones and no interaction among the light particles.

The collision rules are the following:

$$V^+ = \frac{M-1}{M+1} V^- + \frac{2}{M+1} v^-, \qquad v^+ = \frac{2M}{M+1} V^- - \frac{M-1}{M+1} v^- \tag{2.1}$$

or

$$\Delta V = V^+ - V^- = -\frac{2}{M+1}(V^- - v^-); \qquad \Delta v = v^+ - v^- = \frac{2M}{M+1}(V^- - v^-)$$

where $V^{\pm}$ and $v^{\pm}$ are the post- (pre-) collision velocities of the colliding Brownian resp. light particle. The most convenient approach is to describe our system as seen from the Brownian particle (the so-called "Münchhausen picture"). In this picture the phase space is

$$\mathfrak{X} = \mathbb{R} \times \Omega = \{\chi = (V, \omega)\colon V \in \mathbb{R},\ \omega = (q_i, v_i)_{i \in I} \in \Omega\}$$

where I is a countably infinite index set, Ω is the set of locally finite countable point systems in $\mathbb{R} \times \mathbb{R}$, V is the velocity of the Brownian particle, and $(q_i, v_i)_{i \in I}$ are the coordinates (relative to the position of the Brownian particle) and the velocities of the light particles. We say that ω is the environment seen by the Brownian particle. Ω is a Polish space endowed with the natural σ-algebra $\mathscr{F}_0$ generated by counting functions on compact sets. The σ-algebra on $\mathfrak{X}$ is $\mathscr{F} = \mathscr{B} \times \mathscr{F}_0$, $\mathscr{B}$ being the Borel algebra on $\mathbb{R}$. The system is distributed according to the Gibbs measure

$$\mu^M(d(V, \omega)) = dF^M(V) \cdot \nu(d\omega)$$

with ν the Poisson measure on $(\Omega, \mathscr{F}_0)$ with intensity $dx\, dF^1(v)$, and

$$dF^M(V) = (M/2\pi)^{1/2} \exp(-MV^2/2)\, dV \qquad (M > 0)$$

Denote by S_t^M the dynamics of the system. The following two facts are assumed to be known:

(a) For each M there exists a set $\mathfrak{X}^M \subset \mathfrak{X}$ of μ^M-measure 1 on which the maps S_t^M are well defined for any $t \in \mathbb{R}$, and $S_{t+s}^M = S_t^M \circ S_s^M$. (The equilibrium dynamics exists with probability 1.)

(b) The group of transformations $S_t^M: \mathfrak{X}^M \to \mathfrak{X}^M$ preserves the measure μ^M. The random variables to be introduced below are defined on different probability spaces, depending on M.

We shall use the notations

$$V(\chi) = V \text{ and } \omega(\chi) = \omega \qquad \text{iff} \quad \chi = (V, \omega) \in \mathfrak{X}$$

$$V_t^M(\chi) = V(S_t^M \chi), \qquad \chi \in \mathfrak{X}^M$$

$$Q_t^M(\chi) = \int_0^t V_s^M(\chi)\, ds, \qquad \chi \in \mathfrak{X}^M$$

Throughout this paper $W_t^{(\sigma)}$ will denote a Wiener process of variance σ^2 with $W_0^{(\sigma)} = 0$, and for brevity we let $W_t = W_t^{(1)}$.

The diffusion process η_t satisfying the stochastic differential equation

$$d\eta_t = -\gamma \eta_t\, dt + D^{1/2}\, dW_t$$

is called an Ornstein–Uhlenbeck (velocity) process. If η_0 is distributed according to the Gaussian law with mean 0 and variance $(2\gamma)^{-1} D$, then η_t is a stationary Gauss–Markov process.

The integral process

$$\xi_t = \int_0^t \eta_s\, ds$$

is called the Ornstein–Uhlenbeck position process. We shall use these processes with the following choice of parameters:

$$\gamma = (4/m)(2/\pi)^{1/2}, \qquad D = (8/m^2)(2/\pi)^{1/2}$$

and we will use the notations $\eta_t^{(m)}$ and $\xi_t^{(m)}$ for them (m is a positive constant). It is worth mentioning that if $\gamma \to \infty$, $D \to \infty$ in such a way that $D\gamma^{-2} \to \sigma^2 \in \mathbb{R}^+$, then the Ornstein–Uhlenbeck position process ξ_t converges in distribution to a Wiener process $W^{(\sigma)}$ (see Ref. 11). Thus,

$$\xi_t^{(m)} \Rightarrow W_t^{(\sigma)} \qquad \text{as} \quad m \to 0 \tag{2.2}$$

with $\sigma^2 = (\pi/8)^{1/2}$.

3. SURVEY OF SOME RESULTS

Our final aim is to give a complete asymptotic description of the random processes

$$Q_{At}^{M(A)}/A^{1/2} \qquad \text{as} \quad A \to \infty$$

Observe that the space-time scaling is the usual diffusion one, which is used, for example, to obtain a Wiener limiting process for random walks. $M(A)$ expresses the dependence of the mass ratio of the Brownian particle versus the light ones on the parameter A figuring in the space-time scaling.

Throughout this paper $f(A) \ll g(A)$ will have the precise meaning $f(A) = o(g(A))$.

Several cases have already been clarified, but the picture is still far from complete. Here we list the most important results following a logical order rather than a chronological one.

A. For $M(A) \equiv 1$, that is, the Brownian particle is identical with the light ones, Harris[7] and Spitzer[15] proved that

$$Q^1_{A\cdot}/A^{1/2} \Rightarrow W^{(\bar\sigma)}, \qquad \bar\sigma^2 = (2/\pi)^{1/2}$$

Throughout this paper, "$\Rightarrow$" stands for weak convergence on $C[0, \infty)$ (or on $D[0, \infty)$), the space of continuous functions (or right continuous ones without a second-order discontinuity) on $[0, \infty)$.

B. For arbitrary fixed mass $M(A) \equiv M$, Sinai and Soloveychik[14] and the present authors[17] showed that

$$\left(\frac{\pi}{8}\right)^{1/2} t = \underline{\sigma}^2 t \leqslant \varliminf_{A\to\infty} \mathbb{E}\left(\frac{Q^M_{At}}{A^{1/2}}\right)^2 \leqslant \bar\sigma^2 t = \left(\frac{2}{\pi}\right)^{1/2} t \tag{3.1}$$

Computer results[12,13] suggest the following picture: for every M

$$\sigma^2_M = \frac{1}{t} \lim_{A\to\infty} \mathbb{E}\left(\frac{Q^M_{At}}{A^{1/2}}\right)^2$$

exists and by (3.1), of course, $\underline{\sigma} \leqslant \sigma_M \leqslant \bar\sigma$. The dependence of σ_M on M is illustrated in Fig. 1.

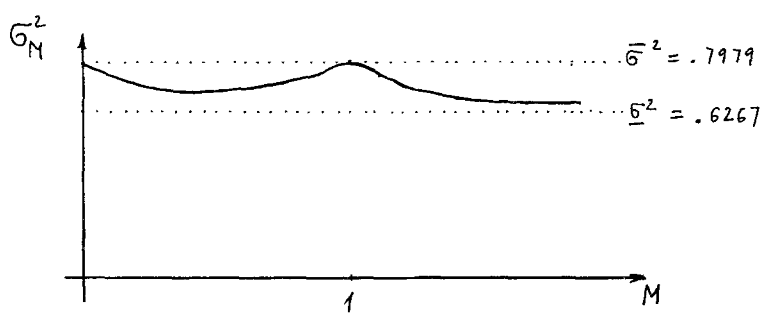

Fig. 1. Computer results for the dependence of the limiting variance on the mass.

C. From the proofs of Ref. 17 it is easy to see that, in (2.3), the upper bound holds for an arbitrary scaling functional $M(A)$, while the lower bound holds whenever $M(A) = o(A)$.

D. For $M(A) = m \cdot A$, $m \in (0, \infty)$, Holley[8] proved that

$$A^{1/2} V_{A\cdot}^{mA} \Rightarrow \eta^{(m)}; \qquad Q_{A\cdot}^{mA}/A^{1/2} \Rightarrow \xi^{(m)}$$

Important Remark. The results B and D can be linked by observing that [cf. (2.2)]

$$\xi^{(m)} \Rightarrow W^{(\sigma)} \qquad \text{as} \quad m \to 0$$

4. THE THEORY

On the basis of the aforementioned results, we expect the following complete asymptotic picture.

1. *Case* $M(A) \to 0$:

$$Q_{A\cdot}^{M(A)}/A^{1/2} \Rightarrow W^{(\bar{\sigma})}$$

2. *Case* $M(A) \equiv M$:

$$Q_{A\cdot}^{M}/A^{1/2} \Rightarrow T^{M}, \qquad \underline{\sigma} \leqslant \sigma_M \leqslant \bar{\sigma}$$

where T^M, $M > 0$, are random processes with stationary increments and with asymptotic variance σ_M^2.

We know that $T^1 = W^{\bar{\sigma}}$, while simulations support that $\sigma_M \to \underline{\sigma}$ as $M \to \infty$ and $\sigma_M \to \bar{\sigma}$ as $M \to 0$ (the result for $M \equiv 1$ was proved in Refs. 7 and 15, while the bounds on the variances were given in Refs. 14 and 17).

3. *Case* $1 \ll M(A) \ll A$:

$$Q_{A\cdot}^{M(A)}/A^{1/2} \Rightarrow W^{(\underline{\sigma})}$$

4. *Case* $M(A) = mA$:

$$Q_{A\cdot}^{M(A)}/A^{1/2} \Rightarrow \xi^{(m)}$$

where $\xi^{(m)}$ is introduced in Section 2. This convergence was proved in Ref. 8. For $m \to 0$, (2.2) holds. For $m \to \infty$, $\xi^{(m)} \Rightarrow 0$.

5. *Case* $M(A) \gg A$:

$$Q_{A\cdot}^{M(A)}/A^{1/2} \Rightarrow 0 \qquad \text{(trivial)}$$

Of course, this case is not trivial if we allow spatial rescalings different from $A^{1/2}$. Indeed, by an application of a theorem of Kurtz (essentially by the same method as that used in Ref. 8), we find the correct asymptotics

$$V_{At}^{M(A)} = \frac{1}{[M(A)]^{1/2}} \{[M(A)]^{1/2} V_0^{M(A)}\}^{1/2} + \frac{A^{1/2}}{M(A)} \xi_t^A$$

where $[M(A)]^{1/2} V_0^{M(A)}$ has a standard Gaussian distribution and ξ_t^A converges in distribution to a Wiener process of variance $4(2/\pi)^{1/2}$. (If $d > 1$, a similar statement holds with a limiting variance depending on the dimension.)

One further step in completing the picture outlined above is the following.

Theorem 1. If $A^{1/2+\varepsilon} \ll M(A) \ll A$ $(\varepsilon > 0)$, then

$$Q_{A\cdot}^{M(A)}/A^{1/2} \Rightarrow W^{(\sigma)}$$

5. SKETCH OF PROOF OF THEOREM 1

The first—and in a sense principal—difficulty in the dynamics of the Brownian particle is the non-Markovian nature of its motion. Indeed, light particles between their first and last collisions with the Brownian one carry information on past collisions in a complicated way. Nonetheless, it is a natural idea to consider a Markov process whose evolution mimics the physical process; this Markov process, of course, disregards recollisions that could spoil its Markovian nature. This Markov version can help both on an intuitive level, to give a feeling for what the mechanical process is like, and on a technical level, if we can construct a good coupling between the mechanical and the Markov processes. To our knowledge, this idea was first used in a rigorous argument by Holley[8] (cf. case D, Section 3) and our proof is also a realization of this strategy.

Let us first construct a family of Markov processes $\tilde{V}_t^M$, $M > 0$, closely related to the mechanical velocity processes V_t^M. In words, the $\tilde{V}_t^M$ are defined as follows: we imagine that the environment is recreated after each collision corresponding to the time-invariant distribution ν. Thus, the Markovian velocity process $\tilde{V}_t^M$ is a pure jump process on $\mathbb{R}$ with jump rates

$$\text{Rate}\left(V \to \frac{M-1}{M+1} V + \frac{2}{M+1} v\right) = \frac{1}{(2\pi)^{1/2}} e^{-v^2/2} |V - v| \, dv$$

In the actual coordinates the jump rates are

$$R^M(x, y)\,dy = \frac{1}{(2\pi)^{1/2}}\left(\frac{M+1}{2}\right)^2 \exp\left[-\frac{1}{2}\left(\frac{M+1}{2}y - \frac{M-1}{2}x\right)^2\right]|x-y|\,dy$$

leading to the formal generator

$$\begin{aligned}(\tilde{G}_M\phi)(x) &= \frac{1}{(2\pi)^{1/2}}\int dy\left(\exp -\frac{y^2}{2}\right)|y-x|\left[\phi\left(\frac{M-1}{M+1}x + \frac{2}{M+1}y\right) - \phi(x)\right] \\ &= \frac{1}{(2\pi)^{1/2}}\left(\frac{M+1}{2}\right)^2 \int dy\left\{\exp\left[-\frac{1}{2}\left(\frac{M+1}{2}y - \frac{M-1}{2}x\right)^2\right]\right\}|y-x| \\ &\quad\times [\phi(y) - \phi(x)] \qquad (5.1)\end{aligned}$$

It is easily seen that, for ϕ and ψ belonging to a sufficiently large class of functions

$$\int dF^M(x)\,\phi(x)(\tilde{G}_M\psi)(x) = \int dF^M(x)(\tilde{G}_M\phi)(x)\,\psi(x)$$

Thus, the Markov processes $\tilde{V}_t^M$ conditioned to the initial distributions $dF^M(x)$ are stationary and reversible. (We shall see soon that they are ergodic, too.)

Now the program consists of two parts:

(i) A study of the induced position processes

$$\tilde{Q}_t^M = \int_0^t \tilde{V}_s^M\,ds$$

(ii) Construction of a good coupling for Q_t^M and $\tilde{Q}_t^M$, i.e., a realization of Q_t^M and $\tilde{Q}_t^M$ on the same probability space that satisfies

$$\frac{1}{A^{1/2}}(Q_{At}^{M(A)} - \tilde{Q}_{At}^{M(A)}) \Rightarrow 0 \quad \text{as} \quad A\to\infty \qquad (5.2)$$

6. THE MARKOVIAN PICTURE

For the approximating Markov process our main result is the following.

Theorem 2. (i) (Fixed masses). For any fixed $M \in (0, \infty)$

$$\tilde{Q}^M_{A\cdot}/A^{1/2} \Rightarrow W^{(\tilde{\sigma}_M)}$$

with

$$\tilde{\sigma}^2_M \geqslant (1 + 1/M)^{1/2} \underline{\sigma}^2$$

and

$$\lim_{M \to \infty} \tilde{\sigma}^2_M = \underline{\sigma}^2$$

(ii) (Sublinearly increasing masses). If $1 \ll M(A) \ll A$, then

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \Rightarrow W^{(\underline{\sigma})}$$

In fact, our methods give the following complete asymptotic characterization of the induced position processes $\tilde{Q}^M_t$ (the reader is encouraged to compare it with the analogous picture formulated for Q^M_t in the preceding section).

1. *Case* $M(A) \to 0$:

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \text{ is not tight}$$

2. *Case* $M(A) \equiv M$:

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \Rightarrow W^{(\tilde{\sigma}_M)}$$

with $\tilde{\sigma}^2_M \sim M^{-1/2}$ for $M \to 0$ and $\tilde{\sigma}^2_M \to \underline{\sigma}^2$ as $M \to \infty$.

3. *Case* $1 \ll M(A) \ll A$:

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \Rightarrow W^{(\underline{\sigma})}$$

4. *Case* $M(A) = mA$:

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \Rightarrow \xi^m$$

5. *Case* $M(A) \gg A$:

$$\tilde{Q}^{M(A)}_{A\cdot}/A^{1/2} \Rightarrow 0$$

Cases 1–3 follow from Theorem 2; Case 4 is proved in Ref. 8; Case 5 is trivial.

It is instructive to compare the pictures for the mechanical versus the Markovian processes (Table I).

Table I

$d=1$	$A^{-1/2}Q_{A\cdot}^{M(A)} \Rightarrow$		$A^{-1/2}\tilde{Q}_{A\cdot}^{M(A)} \Rightarrow$
	Expected	Proved[a]	Proved[a]
$M(A) \ll 1$	$W^{\bar{\sigma}}$	??	Explodes
$M(A) \equiv M$	T^M $\sigma_M \to \underline{\sigma}$ if $M \to \infty$	?? Only known[14,17]: $\underline{\sigma} \leqslant \sigma_M \leqslant \bar{\sigma}$	$W^{\tilde{\sigma}_M}$ Also known: $\tilde{\sigma}_M^2 \sim M^{-1/2}$ $(M \to 0)$ $\tilde{\sigma}_M^2 \to \underline{\sigma}^2$ $(M \to \infty)$ $\tilde{\sigma}_M^2 \geqslant (1+1/M)^{1/2} \sigma_M^2$
$1 \ll M(A) \ll A$	$W^{\underline{\sigma}}$	?? Known if $M(A) \gg A^{1/2+\varepsilon}$	$W^{\underline{\sigma}}$
$M(A) = mA$	ξ^m	Ref. 8	ξ^m (Ref. 8)
$M(A) \gg A$	0	Trivial	0

[a] Proved results without a reference are contained in Ref. 18.

Denote $\hat{V}_t^M = M^{1/2}\tilde{V}_{Mt}^M$. Then

$$A^{-1/2}\tilde{Q}_{At}^M = (A/M)^{-1/2} \int_0^{tA/M} \hat{V}_s^M \, ds$$

are additive functionals of stationary, reversible Markov processes with a common invariant measure, namely the standard Gaussian one. Consider now the Hilbert spaces

$$\mathscr{H}^M = L_2(\mathbb{R}, (M/2\pi)^{1/2} e^{-Mx^2/2})$$

and introduce the unitary isomorphisms $U^M: \mathscr{H}^M \to \mathscr{H}^1$ by $(U^M\phi)(x) = \Phi(x/M^{1/2})$. Then $\tilde{G}_M$ acts in $\mathscr{H}^M$, while the generator of $\hat{V}_t^M$ is MG_M acting in $\mathscr{H}^1$, where $G_M = U^M \tilde{G}_M (U^M)^{-1}$. The proof of Theorem 2 relies upon a lemma ensuring the uniform (in M) ergodicity of the Markov processes $\hat{V}_t^M$ in a rather strong sense.

Gap Lemma. If $1 \leqslant M < \infty$, then

$$\operatorname{Spec}(MG_M) \cap \left(-\frac{M}{M+1} \left(\frac{8}{\pi}\right)^{1/2}, 0 \right) = \varnothing$$

This uniform ergodicity is combined with a martingale approximation, which is a useful tool to obtain central limit theorems for additive functionals of Markov processes.[4,6,9] In fact, part (i) of Theorem 2 follows directly from Ref. 6 with the exact limiting variance

$$\tilde{\sigma}_M^2 = -2(\Phi, (MG_M)^{-1}\Phi)_{\mathscr{H}^1}, \qquad \text{where} \quad \Phi(x) = x$$

Part (ii), however, requires additional arguments, since we have a double array of Markov processes in this case.

7. COUPLING

Coupling Lemma. For every M the processes V_t^M and $\tilde{V}_t^M$ can be defined in the same probability space (Y^M, P^M) in such a way that:

(i) The distribution of V_t^M $(\tilde{V}_t^M)$ coincides with that of the mechanical (Markovian) velocity process.

(ii) If $M(A) \gg A^{1/2+\varepsilon}$, then for any $\eta > 0$ and $t > 0$

$$P^{M(A)}\left(\left\{A^{-1/2}\int_0^{At} ds\, |V_s^{M(A)} - \tilde{V}_s^{M(A)}| > \eta\right\}\right) \to 0$$

Modulo some slight deviations, our coupling is the same as the one used in Ref. 5; therefore, we do not go into details here. The main difference is in proving (ii), i.e., that the coupling is good.

The following inequalities are evident:

$$\begin{aligned} P^{M(A)}&\left(\left\{\sup_{0\leqslant t'\leqslant t} A^{-1/2}\left|\int_0^{At'} ds\,(V_s^{M(A)} - \tilde{V}_s^{M(A)})\right| > \eta\right\}\right)\\ &\leqslant P^{M(A)}\left(\left\{A^{-1/2}\int_0^{At} ds\,|V_s^{M(A)} - \tilde{V}_s^{M(A)}| > \eta\right\}\right)\\ &\leqslant P^{M(A)}(\{\sup_{0\leqslant t'\leqslant t} tA^{1/2}\,|V_{At'}^{M(A)} - \tilde{V}_{At'}^{M(A)}| > \eta\}) \end{aligned} \tag{7.1}$$

To have a "good coupling," i.e., to have (5.2), it is necessary and sufficient to show that the smallest probability of this chain converges to zero. Unfortunately, with our present method, we are only able to handle the second expression. As one can find after understanding the dynamics of the proof, the Coupling Lemma is sharp in this context; that is, if $M(A) = O(A^{1/2})$, then the assertion of the lemma does not hold. (But the coupling may still be "good"—and we expect that actually it is "good" for $M(A) \gg 1$!) On the other hand, for $M(A) \gg A^{3/5+\varepsilon}$, we are able to prove that the largest probability above still converges to zero (this fact may be useful if one also wants to bound the decay of the velocity autocorrelation function).

The fundamental idea in the construction of a good coupling is that the processes V_t^M and $\tilde{V}_t^M$ should suffer as many joint collisions with light particles as possible. Our main observations are:

(a) These joint collisions have a contractive effect for the difference of the velocities, as is easily seen from the collision equations (similar ideas were heavily used in Ref. 1).

(b) To obtain a good bound for the L_1 deviation of the velocity processes, one does not need to add up the effects of all collisions, but it is sufficient to use certain integrals of exponentially decaying functions incorporating the contractive effects of (a).

8. PROBLEMS

8.1. One-Dimensional Case

The question marks in Table I represent the main areas of problems. We consider them in more detail.

8.1.1. Case $M(A) \ll 1$. Since the conjecture $A^{-1/2} Q_{A\cdot}^{M(A)} \Rightarrow W^{\bar{\sigma}}$, if $A \to \infty$ and $M(A) \to 0$, comes from a simple intuitive perturbative argument, it is reasonable to first look for a rigorous perturbative proof to show that

$$\lim_{A \to \infty, M(A) \to 0} A^{-1} \operatorname{Var} Q_A^{M(A)} = \bar{\sigma}^2$$

8.1.2. Case $M(A) \equiv M$. (a) It is an extremely intruguing question whether, in general, T^M is a Wiener process or not. Computer results by Sinai's group[13] suggest that, for general M, it is not. On the other hand, other computer results[12] also gave the $c(M) \circ t^{-3}$ asymptotic decay of the velocity autocorrelation $EV_0^M V_t^M$ that had been known for the Wiener case $M \equiv 1$.[10]

(b) The solution of the previous problem seems hopeless at present, since, for example, no good coupling exists; therefore we expect further progress in more modest directions, e.g., our estimates for $\tilde{\sigma}_M$ may be of some use in bounding σ_M for large M and to show first that $\sigma_M < \bar{\sigma}$ if M is large. It is more difficult to show that the relations $\lim_{M \to \infty} \sigma_M = \underline{\sigma}$, $\lim_{M \to 1} \sigma_M = \lim_{M \to 0} \sigma_M = \bar{\sigma}$ suggested by the numerical results hold.

8.1.3. Case $1 \ll M(A) \ll A^{1/2+\epsilon}$. At present we have no device to catch the cancellations that occur in the smallest probability of (7.1) by symmetry reasons. Here a clever argument may help, and to expect further progress is realistic.

Table II

$d>1$	$A^{-1/2}Q_{A\cdot}^{M(A)} \Rightarrow$		$A^{-1/2}\tilde{Q}_{A\cdot}^{M(A)} \Rightarrow$
	Expected	Proved[a]	Proved[a]
$M(A) \ll 1$	??	??	Explodes
$M(A) \equiv M$	$W^{\sigma_{d,R}^M}$	??	$W^{\tilde{\sigma}_{d,R}^M}$
	$\sigma_{d,R}^M \geqslant \underline{\sigma}_{d,R}$	??	—
$1 \ll M(A) \ll A$	$W^{\underline{\sigma}_{d,R}}$	?? Only if $M(A) \gg A^{1/2+\varepsilon}$	$W^{\underline{\sigma}_{d,R}}$
$M(A) = mA$	$\xi_{d,R}^m$	Ref. 5	$\xi_{d,R}^m$ (Ref. 5)
$M(A) \gg A$	0	Trivial	0 trivial

[a] Proved results without a reference are contained in Ref. 18.

8.2. Multidimensional Case

Now an additional nontrivial parameter appears: the radius $R = R(A)$ of the spherical Brownian particle. For simplicity, we suppose $R(A) \equiv R$ and then the possibilities are the same as those in the 1D situation. Before formulating the questions, we again compile the possibilities (Table II).

A Markov approximation $\tilde{Q}_t^M$ was defined in Ref. 5 and the arguments (the Gap Lemma and its consequences) of Ref. 18 extend to this case, too. Moreover, for $M(A) = mA$, Dürr *et al.*[5] constructed a good coupling to show that the limiting process is a d-dimensional Ornstein–Uhlenbeck one. The method of our aforementioned paper also gives a complete analog of Theorem 1 with the limiting variance $\underline{\sigma}_{d,R}^2 = R^{1-d}d(\pi/8)$.

8.2.1. Case $M(A) \equiv M$. We expect a limiting Wiener process of variance $(\sigma_{d,R}^M)^2$ for any $d \geqslant 2$, though a proof seems realistic for $d \geqslant 5$ only, since then the memory decays sufficiently strongly so as to give hope that an inductive argument works.[2,3]

The conjectured inequality is an analog of the 1D one, but it is a further question whether a similar upper bound can be expected at all.

8.2.2. Case $1 \ll M(A) \ll A$. Same as in the 1D situation.

8.2.3. Asymptotic Independence of Two Brownian Particles. Suppose that two extended spherical particles of equal masses M move in an ideal gas and collide elastically with each other and with the gas particles. Here the system cannot be considered in equilibrium, but we expect that for reasonable initial measures (e.g., the Brownian particles start from $Q_1^M(0)$ and $Q_2^M(0)$; $|Q_1^M(0)-Q_2^M(0)|>2R$ with Maxwellian velocities and the measure of the light particles is Gibbsian outside the domains occupied by the spheres) and for $M(A) \gg 1$ the rescaled trajectories $A^{-1/2}[Q_j^{M(A)}(At)-Q_j^{M(A)}(0)]$, $j=1, 2$, behave asymptotically in the same way as independent copies of processes prescribed by Table II.

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